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Sylvie Galichet [1,2] **Didier Dubois**[1] **Henri Prade**[1]

[1] IRIT, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4

[2] LISTIC, Université de Savoie, 41 avenue de la plaine, BP806, 74016 Annecy Cedex

galichet@univ-savoie.fr dubois@irit.fr prade@irit.fr

Abstract

Functional laws may be known only at a finite number of points, and then the function can be completed by interpolation techniques obeying some smoothness conditions. We rather propose here to specify constraints by means of gradual rules for delimiting areas where the function may lie between known points. Such an approach results in an imprecise interpolation graph whose shape is controlled by tuning the fuzziness attached to the reference points. However, the graph so-built is still crisp, which means that different possible paths between the interpolation points cannot be distinguished according to their plausibility. The paper discusses a method for introducing membership degrees inside the interpolation graph. The developed formalism relies on the use of weighted nested graphs. It amounts to handling level 2 gradual rules for specifying a family of flexible constraints on the reference points. The proposed approach is compared with the one of extending gradual rules for dealing with type 2 fuzzy reference points.

Keywords: Gradual rules, Fuzzy interpolation, Level 2 fuzzy sets

1 Introduction

The main purpose of this paper is to further investigate the interest of gradual rules [5] for modelling interpolative reasoning. What is supposed to be known, in a precise or in an imprecise way, is the behaviour of a system at some points or in some areas, the problem being to interpolate between these regions. The proposed rule-based approach is an alternative to works based on fuzzy polynomial [10] or fuzzy spline interpolation [9], [1], which rely on fuzzy-valued functions. Using such techniques, fuzzy interpolation depends on the analytical form of the interpolant (linear, polynomial, spline-based).

Using a rule-based formalism, we are no longer looking for a function, possibly fuzzy, but for a relation linking input variables to output variables, still constrained by the points which govern the interpolation. Actually, each constraint is expressed by a rule that encapsulates the underlying reference point. This principle was developed in [7], then applied to imprecise modeling for the classification of time series [8]. In these first attempts, the built interpolation graphs are imprecise but still crisp. Actually, the fuzziness introduced for modeling closeness with respect to precise interpolation points is not present in the interpolation graph. This paper presents a refinement of the rule-based representation that enables the handling of fuzzy interpolation graphs. The proposed method consists in implementing level 2 gradual rules.

The paper, after some brief background on gradual rules, discusses interpolation between precisely known points. Then, the building of nested interpolative graphs is addressed. Their weighting allows the definition of fuzzy graphs as level 2 fuzzy sets of crisp graphs. Finally, it is shown that similar results can be obtained using a type 2 fuzzy set-based approach.

2 Interpolation and gradual rules

In the one-dimensional input case considered in the paper, the input-output relation is represented by its graph Γ defined on the Cartesian product $X \times Z$ (where X is the input domain, and Z the output domain). The idea of imprecise interpolation suggested above is based on constraints to be satisfied, namely the postulate that the results of the interpolation should agree with the interpolation points. These constraints should be expressed in order to define the graph Γ of the relation on $X \times Z$.

We consider the case of precise interpolation points P_i with coordinates (x_i, z_i), $i = 1, ..., n$. Then the relation Γ should satisfy:

$$\Gamma(x_i, z_i) = 1,$$
$$\forall\ z \neq z_i \in Z, \quad \Gamma(x_i, z) = 0,$$

for $i = 1, ..., n$. Without any further constraint on the nature of the interpolation, we only have:

$$\forall\ x \neq x_i \in X,\ \forall\ z \in Z,\ \ \Gamma(x, z) = 1.$$

Thus each interpolation point induces the constraint "If $x = x_i$ then $z = z_i$", represented by $(x = x_i) \rightarrow (z = z_i)$ where $\rightarrow$ is the material implication. The relation Γ is thus obtained as the conjunction:

$$\Gamma(x, z) = \wedge_{\ i = 1, .., n}\ \ (x = x_i) \rightarrow (z = z_i). \qquad (1)$$

This relation is extremely imprecise since there is no constraint at all outside the interpolation points. The absence of a choice of a precise type of interpolation function should be alleviated by the use of fuzzy rules in order to express constraints in the vicinity of the interpolation points. The idea is to use rules of the form "the closer x is to x_i, the closer z is to z_i" [5]. The extension to gradual rules of equation (1) provides the following expression for the graph Γ:

$$\Gamma(x, z) = \min_{\ i = 1, .., n}\ \mu_{\text{close to } x_i}(x) \rightarrow \mu_{\text{close to } z_i}(z) \qquad (2)$$

where $\rightarrow$ represents Rescher-Gaines implication, i.e. $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = 0$ if $a > b$, and the value $\mu_{\text{close to } x_i}(x)$ is the degree of truth of the proposition "x is close to x_i".

Two comments on equation (2) are worth stating. First, the principle underlying the rules is the one at work in analogical or case-based reasoning and (2) is interpreting this principle as a constraint (as opposed to a weaker interpretation leading to Mamdani-like fuzzy systems, see [3]). Moreover, (2) embeds interpolation in a purely logical setting (see [2]) which does not require a defuzzification step.

We have now to define what is meant by "close to". Let A_i denote the fuzzy set of values close to x_i. It is natural to set $\mu_{A_i}(x) = 1$ if $x = x_i$ and to assume that the membership degree to A_i decreases on each side of x_i with the distance to x_i. The simplest solution consists in choosing triangular fuzzy sets with a support denoted by $[x_i^-, x_i^+]$. In a similar way, the closeness to z_i will be modelled by a triangular fuzzy set B_i with modal value z_i and support $[z_i^-, z_i^+]$. Then the interpolation relation only depends on $4n$ parameters x_i^-, x_i^+, z_i^-, z_i^+ for n interpolation points.

We might think of using a strong fuzzy partition both for X and Z (i.e. $\forall x \in X$, $\Sigma_{i=1, ..., n}\ \mu_{A_i}(x) = 1$ and $\forall z \in Z$, $\Sigma_{i=1, ..., n}\ \mu_{B_i}(z) = 1$), with triangular fuzzy sets A_i and B_i. In this case, as shown in [5] and [7], gradual rules lead to a precise and linear interpolation, as pictured in Figure 1 with 3 interpolation points.

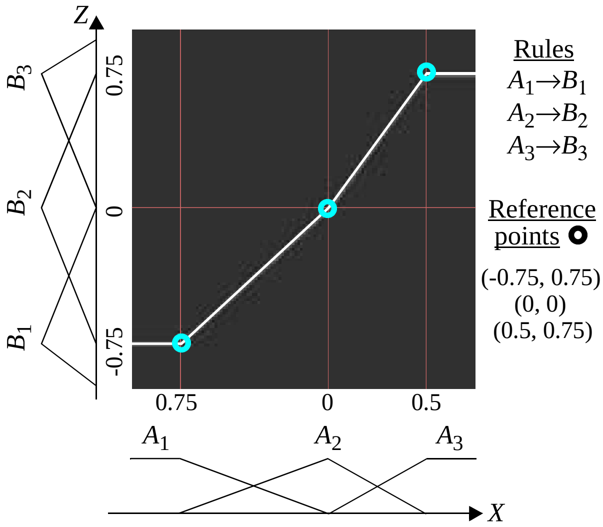

Figure 1: Linear interpolation

The interpolation graph becomes more imprecise when the fuzzy reference points define more permissive constraints. It amounts to modifying their supports while ensuring some tuning conditions. In particular, the graph should remain connected in order to guarantee that any feasible input is associated with an output value. This means that conflict should be avoided when two rules are simultaneously fired, i.e. when $x \in [x_{i+1}^-, x_i^+]$. In other words, coherence conditions for the set of fuzzy rules should be satisfied [6]. In [7], additional conditions are exhibited so as to shape the interpolation graph. Actually, piecewise graphs based on 4-sided areas (see figure 2) can be obtained using a suitable tuning of the fuzzy closeness relation with respect to reference points.

Looking at figure 2, one may be disappointed that the fuzziness introduced in the closeness relations is no more present in the interpolation graph. The next section is devoted to the issue of introducing membership degrees in the 4-sided areas while keeping their support unchanged. Such an appoach is motivated by the need to evaluate the relative merits of different possible paths between the reference points.

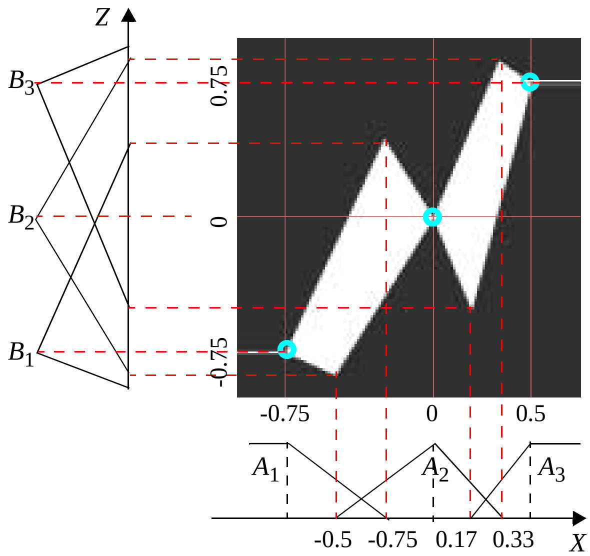

Figure 2: Quadrangle-based interpolation graph

3 Interpolative fuzzy graph

According to equation (2), it is obvious that using a crisp implication for defining the graph necessarily results into a crisp graph. Then, the most intuitive strategy for keeping memberhip degrees in the interpolative graph consists in replacing Rescher-Gaines implication by another one. In order to be in accordance with the semantics of gradual rules, only residuated implications can be used, which means that:

$$a \to b = 1 \text{ if } a \leq b \text{ and } a \to b = \alpha \text{ if } a > b \qquad (3)$$

where α depends on the chosen implication. From equations (2) and (3), it is obvious that the core of the graph Γ (white areas in figure 2) does not depend on the chosen implication. Actually, only values outside the core area (black areas with zero membership grade in figure 2) are concerned with the implication choice. Thus, our aim when introducing membership degrees in the 4-sided areas cannot be achieved simply by choosing a suitable fuzzy implication. As an alternative, an approach based on level 2 gradual rules (still implemented using Rescher Gaines implication) is proposed.

3.1 Nested graph family

According to section 2, it is clear that given a set of rules, i.e. a set of reference points, a collection of crisp graphs is obtained by varying the support parameters of the A_i's and/or the B_i's. Moreover, inclusion properties between the built graphs can be exhibited for controlling the variation of the supports as expressed by the following statements.

P1: If $A_i \subseteq A_i^*$, $i=1, ..., n$, then $\Gamma^* \subseteq \Gamma$, where Γ and Γ^* are the graphs associated with rules $A_i \to B_i$ and $A_i^* \to B_i$ respectively.

The proof of P1 is immediate. Indeed, $(x, z) \in \Gamma^*$ means that $\forall i$, $A_i^*(x) \leq B_i(z)$. According to the assumption that $\forall i$, $A_i \subseteq A_i^*$, it follows that $\forall i$, $A_i(x) \leq B_i(z)$ which results in $(x, z) \in \Gamma$.

P2: If $B_i^* \subseteq B_i$, $i=1, ..., n$, then $\Gamma^* \subseteq \Gamma$, where Γ and Γ^* are now the graphs associated with rules $A_i \to B_i$ and $A_i \to B_i^*$ respectively.

The proof of P2 is also immediate. Indeed, $(x, z) \in \Gamma^*$ if and only if $\forall i$, $A_i(x) \leq B_i^*(z)$. Since $\forall i$, $B_i^* \subseteq B_i$, it follows that $\forall i$, $A_i(x) \leq B_i(z)$, i.e. $(x, z) \in \Gamma$.

The combination of P1 and P2 leads to:

P3: If $A_i \subseteq A_i^*$ and $B_i^* \subseteq B_i$, $i=1, ..., n$, then $\Gamma^* \subseteq \Gamma$, where Γ and Γ^* are the graphs associated with rules $A_i \to B_i$ and $A_i^* \to B_i^*$ respectively.

Thus, according to the above inclusion properties, it is possible to design a family of nested graphs simply by building collections of nested fuzzy subsets on X and Z. Indeed, denote $\{A_i^\lambda, \lambda \in [0,1]\}$ a family of fuzzy subsets on X such that $A_i^{\lambda'} \subseteq A_i^\lambda$ if $\lambda \geq \lambda'$ and $\{B_i^\lambda, \lambda \in [0,1]\}$ a family of fuzzy subsets on Z such that $B_i^\lambda \subseteq B_i^{\lambda'}$ if $\lambda \geq \lambda'$. The graph family associated with rules $A_i^\lambda \to B_i^\lambda$, $\lambda \in [0,1]$, guarantees that $\Gamma^\lambda \subseteq \Gamma^{\lambda'}$ if $\lambda \geq \lambda'$. Actually, such a construction of nested graphs simply expresses that implicative graphs increase in the sense of inclusion when underlying constraints become more permissive. Indeed, more permissive rules are obtained either by restricting input conditions further, or by enlarging output fuzzy sets.

Using a convex linear combination of fuzzy intervals enables the automatic construction of a collection of nested fuzzy subsets ranging from the lower bound of the family to the upper one. Applying such a technique results in the following equation:

$$A_i^\lambda = (1-\lambda)A_i^0 \oplus \lambda A_i^1,\ \lambda \in [0,1],\ i=1, .. ,n \qquad (4)$$

where A_i^0 and A_i^1, such that $A_i^0 \subseteq A_i^1$, are the lower and upper bounds of the family and $\oplus$ denotes the extended sum of fuzzy numbers.

In the same way, nested output fuzzy subsets can be built according to:

$$B_i^{\lambda} = (1-\lambda)B_i^0 \oplus \lambda B_i^1,\ \lambda \in [0,1],\ i=1, .., n \qquad (5)$$

where B_i^0 and B_i^1, such that $B_i^1 \subseteq B_i^0$, are the upper and lower bounds of the family. It should be noted that the inclusion ordering of the B_i^{λ} for increasing λ is the converse of the one of the A_i^{λ}, due to opposite behaviors with respect to graph inclusion.

Using such fuzzy subset families (see figure 3) results in the following graph inclusions:

$$\Gamma^1 \subseteq \Gamma^{\lambda} \subseteq \Gamma^{\lambda'} \subseteq \Gamma^0,\ \lambda,\lambda' \in [0,1] \text{ and } \lambda \geq \lambda'. \qquad (6)$$

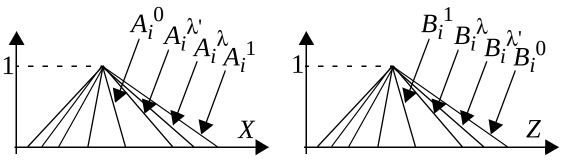

Figure 3: Nested fuzzy subsets ($\lambda > \lambda'$)

Another interesting point is that the 4-sided shape is shared by all nested graphs provided that the lower and upper graphs are themselves quadrangle-shaped.

3.2 Fuzzy graph

According to the previous section, the construction of indexed nested graphs can be easily handled from the knowledge of lower and upper graphs. Now, given the above family $\{\Gamma^{\lambda}, \lambda \in [0,1]\}$, there is a unique fuzzy set F whose λ–cuts F_{λ} are precisely Γ^{λ} for each $\lambda \in [0,1]$. This fuzzy set is built using the standard representation theorem [11], that is:

$$\mu_F(x, z) = \sup_{\lambda \in [0,1]} \min (\lambda, \Gamma^{\lambda}(x, z)) \qquad (7)$$

It should be noticed that equation (7) is based on the equality $F_{\lambda} = \Gamma^{\lambda}$ in which two different meanings of λ are involved depending on the used notation. When λ is used as a subscript, it should be interpreted as a level-cut. On the other hand, the corresponding superscript is relative to the index of an element of a family of nested subsets.

According to formulation (7), the reconstructed F is finally a classical fuzzy graph defined on X x Z. Another interpretation consists in viewing F as a fuzzy set of crisp graphs, that is as a level 2 fuzzy set [12]. In this case, F is represented as:

$$F = \int_{\lambda \in [0,1]} \lambda / \Gamma^{\lambda} \qquad (8)$$

according to Zadeh's notation where the integral sign stands for the union of the fuzzy singletons $\lambda / \Gamma^{\lambda}$.

Figure 4 plots the fuzzy graph obtained when the lower graph Γ^1 is precise and piecewise linear (as in figure 1) and the upper graph Γ^0 has the quadrangle-based shape of figure 2.

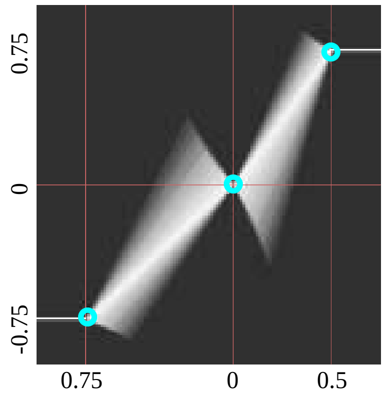

Figure 4: Fuzzy graph obtained by a set of gradual rules

According to Figure 3, families $\{A_i^{\lambda}, \lambda \in [0,1]\}$ and $\{B_i^{\lambda}, \lambda \in [0,1]\}$, $i=1, ..., n$, can also be viewed as type 2 fuzzy subsets, i.e. fuzzy sets with fuzzy membership grades [12]. In this framework, one may wonder if extending the Rescher-Gaines implication to fuzzy set-valued arguments would be compatible with equation (7). Let us examine this approach a little further in the case of a single rule, i.e. n=1.

Let $\tilde{A}$ and $\tilde{B}$ be type 2 fuzzy subsets on X and Z defined according to nested families $\{A^{\lambda}, \lambda \in [0,1]\}$ and $\{B^{\lambda}, \lambda \in [0,1]\}$.

It follows that membership degrees in $\tilde{A}$ and $\tilde{B}$ are now fuzzy subsets on [0,1], that is:

$$\mu_{\tilde{A}}(x) = \tilde{a}(x) \text{ and } \mu_{\tilde{B}}(z) = \tilde{b}(z), \qquad (9)$$

where the membership functions of $\tilde{a}(x)$ and $\tilde{b}(z)$ can be deduced from the construction of $\tilde{A}$ and $\tilde{B}$. When x and z belong to the support of A^0 and B^1, the membership functions obtained for $\tilde{a}(x)$ and $\tilde{b}(z)$ are given in Figure 5.

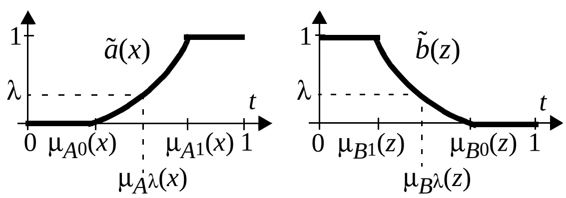

Figure 5: Fuzzy membership degrees $\tilde{a}(x)$ and $\tilde{b}(z)$

Let f be the Rescher-Gaines implication, that is the mapping from [0,1]x[0,1] to {0,1} defined by $f(a,b) = a \rightarrow b$. Extending f to the fuzzy set-valued arguments $\tilde{a}(x)$ and $\tilde{b}(z)$ leads to:

$$\mu_{f(\tilde{a}(x),\, \tilde{b}(z))}(t) = \sup\{\lambda \in [0,1]: t \in f(\tilde{a}(x)_\lambda, \tilde{b}(z)_\lambda)\} \quad (10)$$

where $t \in \{0,1\}$ and $\tilde{a}(x)_\lambda$ (resp. $\tilde{b}(z)_\lambda$) denotes the λ–cut of $\tilde{a}(x)$ (resp. $\tilde{b}(z)$). Since $\tilde{a}(x)_\lambda = [\mu_{A^\lambda}(x), \mu_{A^1}(x)]$ and $\tilde{b}(z)_\lambda = [\mu_{B^1}(z), \mu_{B^\lambda}(z)]$ (see Figure 5), it follows that:

$$\mu_{f(\tilde{a}(x),\, \tilde{b}(z))}(1) = \sup\{\lambda \in [0,1]: \mu_{A^\lambda}(x) \leq \mu_{B^\lambda}(z)\} \quad (11)$$

and

$$\mu_{f(\tilde{a}(x),\, \tilde{b}(z))}(0) = 1 \text{ if } \mu_{A^1}(x) > \mu_{B^1}(z) \quad (12)$$
$$= 0 \text{ otherwise.} \quad (13)$$

From equations (11) and (7), it can be concluded that:

$$\mu_F(x, z) = \mu_{f(\tilde{a}(x),\, \tilde{b}(z))}(1) \quad (14)$$

which means that the fuzzy graph F can be obtained from a type 2 reasoning by considering the “true” case only. The information provided by the “false” case is quite poorer since the degree of membership of 0 to $f(\tilde{a}(x), \tilde{b}(z))$ simply defines the complement of the core of F, i.e. this degree is 1 if $(x, z) \notin \text{core}(F)$ and 0 otherwise.

Links between type 2 and level 2 interpretations are illustrated on figure 6 for a single gradual rule.

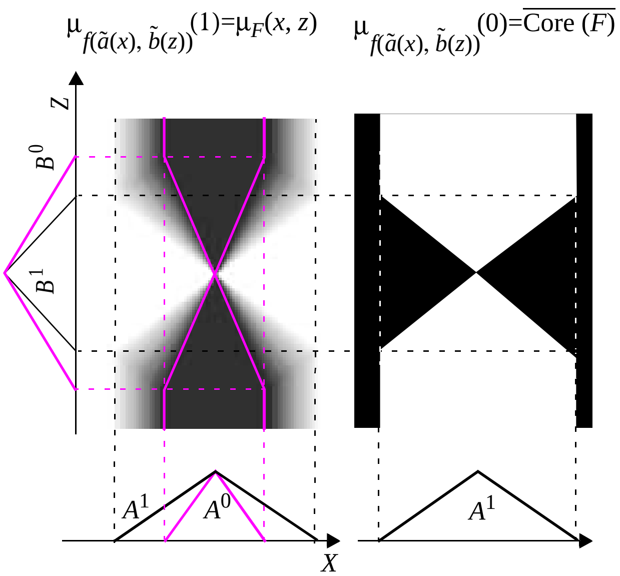

Figure 6: Type 2-based construction

Further computations are still necessary for combining several implicative rules in the case of a type 2 interpretation. Actually, even if both approaches (level 2 and type 2 constructions) lead to similar fuzzy graphs, it is more convenient to adopt the level 2 interpretation.

The proposed approach based on (7) or (14) provides a constructive method for deriving a genuine fuzzy implication from a set of gradual rules. That it is a genuine implication can be checked by verifying the following properties:

if $\mu_{A^1}(x) \geq \mu_{A^1}(x^*)$ then $\mu_F(x, z) \leq \mu_F(x^*, z)$
if $\mu_{B^0}(z) \geq \mu_{B^0}(z^*)$ then $\mu_F(x, z) \geq \mu_F(x, z^*)$

Moreover,

if $\mu_{A^1}(x) = 1$ then $\mu_F(x, z) = 0$ when $\mu_{B^0}(z) \neq 1$
if $\mu_{A^1}(x) = 0$ then $\mu_F(x, z) = 1$
if $\mu_{B^0}(z) = 1$ then $\mu_F(x, z) = 1$ (identity principle).

It is interesting to compare our construction to the one in [4]. This paper establishes results under which fuzzy implications can be decomposed as convex sums of crisp rules. It assumes a finite number of membership grades. Under certain mild conditions, this decomposition involves a nested family of gradual rules of the form $m_i(A) \rightarrow B$ where $\{m_i\}$ is a family of modifiers affecting the condition part only. In the present paper, both conditions and conclusions are varied.

4 Conclusion

This paper illustrates how fuzzy interpolation graphs can be obtained from gradual rules. The easiest strategy consists in building nested crisp graphs whose weighting allows the construction of the final fuzzy graph in the form of a level 2 graph.

This method is compatible with a type 2 interpretation which applies the extension principle to the Rescher-Gaines implication with fuzzy set-valued arguments.

An interesting point for further investigation concerns the interpretation of the interpolative crisp graphs in terms of some properties of the underlying functions, especially their derivatives. If such links could be exhibited, they would provide a theoretical framework for the choice of the lower and upper graphs that are used for the building of the fuzzy interpolative graphs.

From a practical point of view, a relevant use of the proposed fuzzy interpolation technique still requires that the multi-input case be developed. In this context, the construction of nested interpolation graphs by means of families of gradual rules with composite antecedents is also a matter of further research.

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